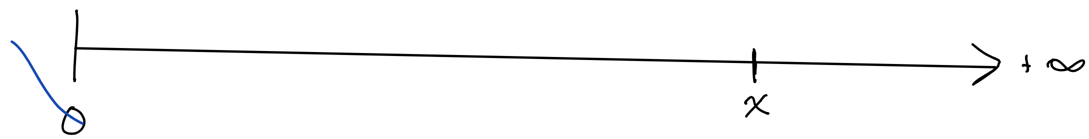
Exponential distribution

Lecture 21

Consider an interval



Suppose that some event occurs randomly throughout the interval, and occurs on average λ times/unit interval, distributed as a Poisson Random Var

Let X be the CRV defined by

$X =$ the length along the interval (from 0, our starting point) until we find an event.

What is $P(X > x)$? $P(X > x)$ is the same as the probability that <u>no</u> events occur in the interval $[0, x]$, and so if

$N_x =$ # of events in $[0, x]$ (and so N_x is Poisson) we have

$$P(X > x) = P(N_x = 0) = e^{-\lambda x} (\lambda x)^0$$

$$= e^{-\lambda x}$$

Thus, $\underbrace{P(X \le x) = 1 - e^{-\lambda x}}_{\text{cummulative dist. function of } X.}$

Recall: to get the pdf from the cdf, we differentiate:

$$P(X=x) = \lambda e^{-\lambda x} \quad \text{for } 0 \le x < \infty.$$

Notice that X depends on the length (x) of the interval, not on where the interval starts (starting at 0 was an arbitrary choice).

If X is an exponential CRV with parameter λ, then

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\,dx = \lim_{t\to\infty} \int_0^t x \lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}.$$

$$V(X) = E[X^2] - (E[X])^2 = 1/\lambda^2.$$

Ex: The average time between buses at your stop is 15 minutes. If the time between buses is exponentially distributed,

a) What is the probability that you get to the stop and a bus arrives in the next 5 minutes?

b) What is the probability that a bus arrives in the next 5 minutes given that you've already waited 20 minutes?

Let X = waiting time for the bus.

What is λ? $15 = 1/\lambda$, so $\lambda = 1/15$.

So pdf: $f(x) = \dfrac{e^{-x/15}}{15}$

cdf: $F(x) = P(X \le x) = 1 - e^{-x/15}$

) $P(X \le 5) = F(5) = 1 - e^{-5/15} = 1 - e^{-1/3}$
$\approx 0.283 \approx 28.3\%$.

b) We use the conditional prob. formula:

$$P(X \le \underbrace{25}_{\text{20 mins + 5 more mins.}} \mid X \ge 20) = \frac{P(20 \le X \le 25)}{P(X \ge 20)}$$

$$= \frac{P(20 \le X \le 25)}{1 - P(X \le 20)} = \frac{F(25) - F(20)}{1 - F(20)}$$

$$= \frac{(\cancel{1} - e^{-25/15}) - (\cancel{1} - e^{-20/15})}{\cancel{1} - (\cancel{1} - e^{-20/15})}$$

$$= \frac{-e^{-25/15} + e^{-20/15}}{e^{-20/15}} = -e^{-5/15} + 1 \approx \underline{0.283}$$ The Same as part a).

This illustrates what we mean when we say X only depends on the length of the interval (in this case, the waiting time) and not the starting point.

We say that X has the Lack of Memory Property if $P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2)$.

(i.e. exponential random vars have this property).

Hazard Rate Function

- Let X be a positive continuous RV.
- we interpret X as the lifetime of some item,

  ex: how long will my computer work before it shuts down.
- Suppose that X has cummulative distribution distribution F and probability density function f.

Defn: The hazard rate (aka failure rate) function $\lambda(t)$ is defined to be

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \frac{f(t)}{1 - F(t)}.$$

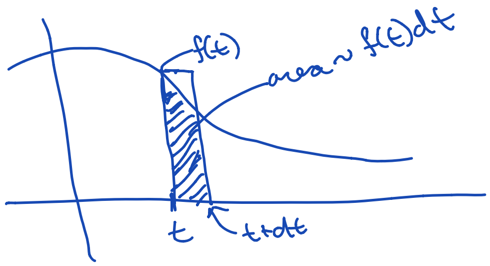
Interpreting $\lambda(t)$: Suppose that our computer (or whatever) has lasted until time t, but will not survive for an additional time dt.

That is, we want to know the propability

$$P(\underbrace{X \in (t, t+dt)}_{\text{end of life between } t \text{ and } t+dt} \mid \underbrace{X > t}_{\text{given that } X \text{ has lasted } t \text{ times.}})$$

using the formula for conditional probability, we have:

$$P(X \in (t, t+dt) \mid X > t) = \frac{P(X \in (t, t+dt) \text{ and } X > t)}{P(X > t)}.$$



$$= \frac{P(X \in (t, t+dt))}{P(X > t)}$$

$$\approx \frac{f(t)dt}{1 - F(t)} = \lambda(t)dt.$$

So $\lambda(t)$ represents the conditional probability that an item of age t will fail.

— Suppose that X is exponential, then by the memoryless property, $\lambda(t)$ is constant.

— we can check this:

$$\lambda(t) = \frac{f(t)}{F(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

More generally, the failure rate function $\lambda(t)$, $t \geq 0$, completely determines the cumulative distribution function F: integrating λ, we have

$$\int_0^t \lambda(s)\,ds = \int_0^t \frac{f(s)}{1-F(s)}\,ds$$

Substituting $u = (1-F(s))$, $du = -f(s)ds$ (since $\frac{d}{ds}F(s) = f(s)$)

$$\int_0^t \lambda(s)ds = \int_{1-F(0)}^{1-F(t)} \frac{-1}{u}\,du = -\log(u)\Big|_{1-F(0)}^{1-F(t)}$$

$$= -\log(1-F(t)) + \underbrace{\log(1-F(0))}_{=0}$$

$F(0)=0$

$$= -\log(1-F(t))$$

Therefore

$$\int_0^t \lambda(s)ds = -\log(1-F(t))$$

So $\boxed{F(t) = 1-\exp\left(-\int_0^t \lambda(s)\,ds\right)}$

For example, if $\lambda(t) = a + bt$, we get

$$F(t) = 1 - e^{-at-bt^2/2}$$

Differentiating,

$$\underbrace{f(t) = (a+bt)e^{-(at+bt^2/2)}, \quad t \geq 0.}$$

If $a=0$, this is the Rayleigh distribution.